

ARITHMETIC PROGRESSIONS IN DISCRETE GROUPS

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ABSTRACT. Given a sequence $\Sigma = \{F_n\}$ of finite subsets in a discrete group Γ , suppose b in Γ is fixed by Σ from right. If a subset Λ of Γ has positive upper density with respect to Σ , then for any positive integer k , there exist $n > 0$ and $a \in \Gamma$ such that $\{b^{jn}a\}_{j=0}^{k-1} \subseteq \Lambda$.

1. INTRODUCTION

In [Sze75], E. Szemerédi proved the following theorem conjectured by P. Erdős and P. Turán [ET36], which generalizes van der Waerden’s theorem [vand27].

Theorem 1.1. [Szemerédi’s theorem]

A set of positive integers with positive upper density contains arbitrarily long arithmetic progressions.

Szemerédi’s original proof is combinatorial and has merits on its own right [Sze69, Sze75]. A good survey is [Tao07].

In 1977, H. Furstenberg found that Szemerédi’s theorem is equivalent to a multiple recurrence theorem, which he called “ergodic Szemerédi theorem”. See [Fur77, Thm. 1.4] and [FKO82, Thm. II].

Via proving his ergodic Szemerédi theorem, Furstenberg gave an ergodic theoretic proof of Szemerédi’s theorem. This is Furstenberg correspondence principle which opens a door for applications of ergodic theory to combinatorial number theory. This is also the main ingredient of the paper.

Theorem 1.2. [Ergodic Szemerédi theorem]

Let (X, \mathcal{B}, ν, T) be a dynamical system consisting of a probability measure space (X, \mathcal{B}, ν) and a measure-preserving transformation $T : X \rightarrow X$. For any positive

Date: April 11, 2017.

2010 Mathematics Subject Classification. Primary 37A45, 11B25, 43A07.

Key words and phrases. Discrete group, arithmetic progression.

integer k , there exists $n \in \mathbb{Z}$ such that

$$\mu\left(\bigcap_{j=1}^k T^{-jn} A\right) > 0$$

whenever $\mu(A) > 0$.

Along this idea, it appear various generalizations of Szemerédi's theorem to \mathbb{Z}^d [FK78, FK91, BL96].

Actually via ergodic Szemerédi theorem, Furstenberg had proved a theorem stronger than Szemerédi's theorem [FKO82, Thm. I].

Theorem 1.3. [Furstenberg's version of Szemerédi's theorem]

A set of positive integers with positive upper Banach density contains arbitrarily long arithmetic progressions.

A subset Λ of positive integers has **positive upper density** if $\limsup_{n \rightarrow \infty} \frac{|\Lambda \cap [1, n]|}{n} > 0$ and has **positive upper Banach density** if $\limsup_{n \rightarrow \infty} \frac{|\Lambda \cap [a_n, b_n]|}{b_n - a_n} > 0$ for a sequence of intervals $\{[a_n, b_n]\}$ with $b_n - a_n \rightarrow \infty$.

The difference between positive upper density and positive upper Banach density cannot be omitted since if one can prove Erdős-Turán conjecture, which says that a subset Λ satisfying that $\sum_{k \in \Lambda} \frac{1}{k} = \infty$ contains arbitrarily long arithmetic progressions, then Szemerédi's theorem immediately follows. However unlike Szemerédi's theorem and the Green-Tao theorem [GT08, Thm. 1.1], Furstenberg's version of Szemerédi's theorem does not follow from trueness of Erdős-Turán conjecture, since a set Λ with positive upper Banach density does not necessarily satisfy that $\sum_{k \in \Lambda} \frac{1}{k} = \infty$.

For instance, let $\Lambda = \bigcup_{n=1}^{\infty} [2^n, 2^n + n)$, then Λ has positive upper Banach density but

$\sum_{k \in \Lambda} \frac{1}{k} \leq \sum_{n=1}^{\infty} \frac{n}{2^n} < \infty$. See [Fur81, Sec. 3.6] for a discussion of comparison between positive upper density and positive upper Banach density.

There also comes a natural question:

what's the reason behind the fact that a set with positive upper density contains arbitrarily long arithmetic progressions?

In this paper, we prove a version of Szemerédi's theorem in a much more general setting and give a partial answer to this question.

Theorem 1.4. Let $\Sigma = \{F_n\}$ be a sequence of finite subsets in a discrete group Γ and suppose b in Γ is fixed by Σ from right (left). If a subset Λ of Γ has positive upper density with respect to Σ , then for any positive integer k , there exist $n > 0$ and $a \in \Gamma$ such that $\{b^{jn}a\}_{j=0}^{k-1}$ ($\{ab^{jn}\}_{j=0}^{k-1}$) is contained in Λ .

Here we say that b in Γ is **fixed by Σ from right (left)** if $\lim_{n \rightarrow \infty} \frac{|bF_n \Delta F_n|}{|F_n|} = 0$ ($\lim_{n \rightarrow \infty} \frac{|F_n b \Delta F_n|}{|F_n|} = 0$).

The **upper density** $\overline{D}_\Sigma(\Lambda)$ of a subset Λ of Γ with respect to Σ is defined by $\limsup_{n \rightarrow \infty} \frac{|F_n \cap \Lambda|}{|F_n|}$ [BBF10].

As a consequence, we have the following generalization of Szemerédi's theorem to amenable groups.

Corollary 1.5. In a subset Λ of an amenable group Γ with positive upper density with respect to a left (right) Følner sequence, for every positive integer k and every b in Γ , there exist a in Γ and a positive integer n such that $\{b^{jn}a\}_{j=1}^k$ ($\{ab^{jn}\}_{j=1}^k$) is contained in Λ .

Moreover if Γ contains \mathbb{Z} as a subgroup, then a subset of Γ with positive upper density with respect to a left (right) Følner sequence contains arbitrarily long left (right) arithmetic progressions.

Here for a, b in Γ , the set $\{b^j a\}_{j=1}^k$ ($\{ab^j\}_{j=1}^k$) of k distinct elements is called a left (right) **arithmetic progression of length k** .

ACKNOWLEDGEMENTS

I thank Hanfeng Li for his illuminating comments to this paper. I got familiar with Furstenberg correspondence principle in a 2012 graduate student seminar in SUNY at Buffalo organized by Bingbing Liang, Yongle Jiang, Yongxiao Lin and myself. I thank them for their kind feedback. The latest version of the paper was carried out during a visit to the Research Center for Operator Algebras in East China Normal University in April 2017. I thank Huaxin Lin for his hospitality and Qin Wang for helpful discussions.

2. SZEMERÉDI'S THEOREM FOR AMENABLE GROUPS

Suppose T is a homeomorphism on a compact metrizable space X . A Borel probability measure ν on X is called **T -invariant** if $\nu(T^{-1}A) = \nu(A)$ for every Borel subset A of X and every s in Γ .

It's well-known that ν is T -invariant if and only if $\nu(T^{-1}f) = \nu(f)$ for every f in $C(X)$. Here $C(X)$ stands for the set of complex-valued continuous functions on X , $\nu(f) = \int_X f(y) d\nu(y)$ and $T^{-1}f(x) = f(T(x))$ for every x in X .

Now we start to prove Theorem 1.4.

Proof. We only need to give a proof for the case that b in Γ is fixed by Σ from right.

If b in Γ is fixed by Σ from left and Λ is a subset of Γ with positive upper density with respect to $\Sigma = \{F_n\}$, then b^{-1} is fixed by $\Sigma^{-1} = \{F_n^{-1}\}$ from left and Λ^{-1} is a subset with positive upper density with respect to Σ^{-1} . The existence of $\{b^{jn}a\}_{j=1}^k$ in Λ^{-1} gives the existence of $\{ab^{jn}\}_{j=1}^k$ in Λ .

Let Γ act on $\{0, 1\}^\Gamma$ by shift, that is, $s \cdot x(t) = x(ts)$ for all s, t in Γ and x in $\{0, 1\}^\Gamma$. Define $A_0 := \{x \in \{0, 1\}^\Gamma | x(e_\Gamma) = 1\}$ and let $\omega = 1_\Lambda$ be the characteristic function of Λ .

Suppose b in Γ is fixed by Σ from left.

Then for every positive integer k , one has

$$\begin{aligned} \exists a \in \Gamma \text{ such that } \{b^j a\}_{j=1}^k \subseteq \Lambda; &\iff \exists a \in \Gamma \text{ such that } \omega(b^j a) = 1 \quad \text{for all } 1 \leq j \leq k; \\ &\iff \exists a \in \Gamma \text{ such that } b^j a \cdot \omega(e_\Gamma) = 1 \quad \text{for all } 1 \leq j \leq k; \\ &\iff \exists a \in \Gamma \text{ such that } \{b^j a \cdot \omega\}_{j=1}^k \subseteq A_0. \end{aligned}$$

Let $X = \overline{\Gamma \cdot \omega}$ be the closure of the orbit of ω in $\{0, 1\}^\Gamma$.

Let $A = A_0 \cap X$, which is a closed subset of X .

It follows that

$$\begin{aligned} \exists a \in \Gamma \text{ such that } \{b^j a \cdot \omega\}_{j=1}^k \subseteq A_0; &\iff \exists a \in \Gamma \text{ such that } \omega \in \bigcap_{j=1}^k (b^j a)^{-1} \cdot A_0; \\ &\iff \exists a \in \Gamma \text{ such that } a \cdot \omega \in \bigcap_{j=1}^k b^{-j} \cdot A_0; \iff \bigcap_{j=1}^k b^{-j} \cdot A_0 \cap \Gamma \cdot \omega \neq \emptyset; \\ (A_0 \text{ is open}) &\implies \bigcap_{j=1}^k b^{-j} \cdot A_0 \text{ is open.} \end{aligned}$$

$$\iff \bigcap_{j=1}^k b^{-j} \cdot A_0 \cap \overline{\Gamma \cdot \omega} \neq \emptyset; \iff \bigcap_{j=1}^k b^{-j} A \neq \emptyset.$$

Next we are going to construct a b -invariant Borel probability measure μ on X such that $\mu(A) > 0$. By Theorem 1.2, this will complete the proof.

Let $\delta_{s \cdot \omega}$ be the Dirac measure at the point $s \cdot \omega$ for s in Γ , and this is a Borel probability measure on X .

Define $\mu_n = \frac{1}{|F_n|} \sum_{t \in F_n} \delta_{t \cdot \omega}$. Let μ be a weak-* limit of μ_n . Without loss of generality, let $\mu = \lim_{n \rightarrow \infty} \mu_n$.

Then the following two claims hold.

- (1) μ is b -invariant.
- (2) $\mu(A) > 0$.

Proof. [Proof of the first claim]

For every continuous function f on X , one has

$$\begin{aligned} \mu(b^{-1} \cdot f) &= \lim_{n \rightarrow \infty} \mu_n(b^{-1} \cdot f) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{t \in F_n} b^{-1} \cdot f(t \cdot \omega) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{t \in F_n} f(bt \cdot \omega) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{t \in bF_n} f(t \cdot \omega) \end{aligned}$$

(b is fixed by Σ from right.)

$$= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{t \in F_n} f(t \cdot \omega) = \mu(f).$$

Hence μ is b -invariant. □

Proof. [Proof of the second claim]

Note that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mu_n(A) &= \limsup_{n \rightarrow \infty} \frac{|\{t \in F_n \mid t \cdot \omega \in A\}|}{|F_n|} \\ &= \limsup_{n \rightarrow \infty} \frac{|\{t \in F_n \mid t \cdot \omega \in A_0\}|}{|F_n|} \\ &= \limsup_{n \rightarrow \infty} \frac{|\{t \in F_n \mid t \cdot \omega(e_\Gamma) = 1\}|}{|F_n|} \\ &= \limsup_{n \rightarrow \infty} \frac{|\{t \in F_n \mid \omega(t) = 1\}|}{|F_n|} \end{aligned}$$

$$\begin{aligned}
&= \limsup_{n \rightarrow \infty} \frac{|F_n \cap \Lambda|}{|F_n|} \\
&= \overline{D}_\Sigma(\Lambda) > 0.
\end{aligned}$$

Since A is a closed subset of X , we have $\mu(A) \geq \limsup_{n \rightarrow \infty} \mu_n(A) > 0$ [Wal82, Sec. 6.1, Remarks (3)]. \square

Applying Theorem 1.2 to the dynamical system (X, μ, b) gives the proof. \square

Remark 2.1. A set Λ has positive upper density with respect to Σ iff it has positive density with respect to a subsequence of Σ , hence without loss of generality, we can just assume that Λ has positive density with respect to a sequence. We implicitly use this fact in the proof of Theorem 1.4.

A sequence $\{F_n\}_{n=1}^\infty$ of finite subsets in a countable discrete group Γ is called a left (right) **Følner sequence** if

$$\lim_{n \rightarrow \infty} \frac{|sF_n \Delta F_n|}{|F_n|} = 0 \quad \left(\lim_{n \rightarrow \infty} \frac{|F_n s \Delta F_n|}{|F_n|} = 0 \right)$$

for every s in Γ . A group Γ having a Følner sequence is called **amenable**.

Remark 2.2. It might happen that except the neutral element, no other element in Γ is fixed by Σ for a sequence Σ . For instance no integer except 0 is fixed by $\Sigma = \{F_n\}_{n=1}^\infty$ for $F_n = \{2, 4, \dots, 2^n\}$. So choices of Σ decide the elements fixed by it.

When Γ is amenable, and one can choose Σ to be a left (right) Følner sequence of Γ . Then every b in Γ is fixed by Σ from right (left).

So Theorem 1.4 gives the following application.

Corollary 2.3. [Arithmetic progressions in amenable groups]

In a subset Λ of an amenable group Γ with positive upper density with respect to a left (right) Følner sequence, for every positive integer k and every b in Γ , there exist a in Γ and a positive integer n such that $\{b^{jn}a\}_{j=1}^k$ ($\{ab^{jn}\}_{j=1}^k$) is contained in Λ .

Moreover if Γ contains \mathbb{Z} as a subgroup, then a subset of Γ with positive upper density with respect to a left (right) Følner sequence contains arbitrarily long left (right) arithmetic progressions.

Proof. If Σ is a left Følner sequence in an amenable group Γ , then every b in Γ is fixed by Σ from right. By Theorem 1.4, the first statement holds.

Since \mathbb{Z} is a subgroup of Γ , there exists an element b of infinite order in Γ . By Theorem 1.4, for every positive integer k , there exist a in Γ and a positive integer n such that Λ contains $\{b^{jn}a\}_{j=1}^k$ ($\{ab^{jn}\}_{j=1}^k$). Since b is of infinite order, the set $\{b^{jn}a\}_{j=1}^k$ ($\{ab^{jn}\}_{j=1}^k$) has k distinct elements. Hence it is a left (right) arithmetic progression of length k . \square

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